Logarithmic Ring Spectra

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Outline

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$$\mathbb{L}_{A|R} := A \otimes_{A_\bullet} \Omega^1_{A_\bullet|R},$$

where $A_\bullet \xrightarrow{\simeq} A$ is a simplicial resolution by free R-algebras.

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of simplicial commutative algebras augmented over A.

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Theorem (Quillen, 1970)

There is a spectral sequence

$$E^2_{p,q} = \pi_p\Big(\bigwedge_A^q \mathbb{L}_{A|R}\Big) \implies \pi_{p+q}\mathrm{HH}^R(A).$$

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A pre-log ring (R, P, β) is a *log ring* if $\widetilde{\beta}$ in the pullback diagram

$$\begin{array}{ccc} \beta^{-1}\mathrm{GL}_1(R) & \xrightarrow{\widetilde{\beta}} & \mathrm{GL}_1(R) \\ \downarrow & & \downarrow \\ P & \xrightarrow{\beta} & (R, \cdot) \end{array}$$

is an isomorphism.

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- There is a factorization

$$(\mathbb{Z}_p, \mathrm{GL}_1(\mathbb{Z}_p)) \to (\mathbb{Z}_p, \langle p \rangle^{\log}) \to (\mathbb{Q}_p, \mathrm{GL}_1(\mathbb{Q}_p)).$$

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where $R[P^{-1}] := R \otimes_{\mathbb{Z}[P]} \mathbb{Z}[P^{\mathrm{gp}}]$ is the *localization* of (R, P).

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The log derivations $\mathrm{Der}_{(R,P)}((A, M), J)$ are corepresented by the A-module of *log Kähler differentials* $\Omega^1_{(A,M)|(R,P)}$.

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Definition (Kato–Saito (2004), Rognes (2009))

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$$(B, K)^{\mathrm{rep}} := (B_K^{\mathrm{rep}}, K^{\mathrm{rep}}) := (B \otimes_{\mathbb{Z}[K]} \mathbb{Z}[K^{\mathrm{rep}}], K^{\mathrm{rep}}) \to (A, M).$$

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Theorem (Kato–Saito (2004))

In the same way that $\Omega^1_{A|R} \cong I/I^2, \quad I := \ker(A \otimes_R A \to A)$, *there is an isomorphism*

$$\Omega^1_{(A,M)|(R,P)} \cong I/I^2, \quad I := \ker((A \otimes_R A)^{\mathrm{rep}}_{(M \oplus_P M)} \to A)$$

Definition (Gabber)

The *log cotangent complex* is the simplicial A-module

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where $(A_\bullet, M_\bullet) \xrightarrow{\simeq} (A, M)$ is a simplicial resolution by free (R, P)-algebras.

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This definition agrees with that of Rognes, and there is a spectral sequence

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- If $x \in R$ is a non-zero divisor, there is a long exact sequence $\cdots \to \mathrm{HH}_*(R/(x)) \to \mathrm{HH}_*(R) \to \mathrm{HH}_*(R, \langle x\rangle) \to \cdots$

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But the $\mathbb{E}_\infty$-space of units $\mathrm{GL}_1(R)$ only sees units in $\pi_0(R)$!

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Solution (Sagave–Schlichtkrull, 2012)

Consider instead $\mathbb{E}_\infty$-spaces graded/augmented over $QS^0 = \Omega^\infty \Sigma^\infty S^0$.

First definition of log ring spectra

Definition

A *pre-log ring spectrum* (R, P, β) consists of

- an $\mathbb{E}_\infty$-ring R;
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Theorem (L., 2020)

These definitions agree with those of Rognes–Sagave–Schlichtkrull.

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In fact, these two assertions are equivalent [L., 2020].

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