Log Hochschild homology via the log diagonal

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The HKR-theorem and étale base-change

Let R be a commutative ring. One can form the *Hochschild homology* $\mathrm{HH}(R)$. This is a simplicial commutative R-algebra. It satisfies $\pi_0\mathrm{HH}(R) \cong R$ and $\pi_1\mathrm{HH}(R) \cong \Omega^1_R$, and so there is a canonical map

$$\Omega^*_R := \Lambda^*_R\Omega^1_R \to \pi_*\mathrm{HH}(R)$$

of graded commutative rings.

Theorem (Hochschild–Kostant–Rosenberg)

This is an isomorphism if R is smooth.

Theorem (Weibel–Geller)

If $R \to A$ is étale, then the canonical map

$$A \otimes_R \mathrm{HH}(R) \to \mathrm{HH}(A)$$

is an equivalence.

The HKR-theorem and étale base-change

Examples of étale maps include $R \to R[x^{-1}]$, $\mathbb{Z} \to \mathbb{Z}[1/2, i]$...

Theorem (Mathew)

If $R \to A$ is an étale map of commutative ring spectra, the canonical map

$$A \otimes_R \mathrm{THH}(R) \to \mathrm{THH}(A)$$

is an equivalence.

The goal of this talk is to relax the smoothness/étaleness hypotheses using techniques from *log geometry*:

- Get a version of HKR for $\mathbb{Z}[x, y]/(xy)$, despite this not being smooth.
- Get a version of étale base-change for $\mathbb{Z}_{(3)} \to \mathbb{Z}_{(3)}[\sqrt{3}]$, despite this not being étale.

Logarithmic rings

Definition (Kato, 1989)

A *pre-logarithmic ring* (R, P, β) consists of

- a commutative monoid P;
- a commutative ring R;
- a map $\beta\colon P \to (R, \cdot)$; equivalently, a map $\bar{\beta}\colon \mathbb{Z}[P] \to R$.

Example

Any ring R admits a *trivial pre-log structure* $(R, \{1\})$.

Example

$(\mathbb{Z}_{(3)}, \langle 3 \rangle)$ is a pre-log ring, with

$$\langle 3 \rangle = \{3^0, 3^1, 3^2, \dots\}$$

and structure map the inclusion.

Logarithmic rings

Definition

A *logarithmic ring* (R, P, β) is a pre-log ring with the property that $\widetilde{\beta}$ in the pullback diagram

$$\begin{array}{ccc} \beta^{-1}\mathrm{GL}_1(R) & \xrightarrow{\widetilde{\beta}} & \mathrm{GL}_1(R) \\ \downarrow & & \downarrow \\ P & \xrightarrow{\beta} & (R, \cdot) \end{array}$$

is an isomorphism.

Example

Trivial log ring $(R, \mathrm{GL}_1(R))$.

Example

$(\mathbb{Z}_{(3)}, \langle 3 \rangle \times \mathrm{GL}_1(\mathbb{Z}_{(3)}))$ is a log ring, with structure map the inclusion.

Logarithmic rings

There is a factorization

$$(\mathbb{Z}_{(3)}, \mathrm{GL}_1(\mathbb{Z}_{(3)})) \to (\mathbb{Z}_{(3)}, \langle 3 \rangle \times \mathrm{GL}_1(\mathbb{Z}_{(3)})) \to (\mathbb{Q}, \mathrm{GL}_1(\mathbb{Q}))$$

in the category of log rings. More generally, for any log ring (R, P) there is a factorization

$$(R, \mathrm{GL}_1(R)) \to (R, P) \to (R[P^{-1}], \mathrm{GL}_1(R[P^{-1}])),$$

where $R[P^{-1}] := R \otimes_{\mathbb{Z}[P]} \mathbb{Z}[P^{\mathrm{gp}}]$.

- Summary: A pre-log structure on a ring R is a map $\beta \colon P \to (R, \cdot)$ from a commutative monoid P, and we think of the resulting object (R, P, β) as an intermediate localization of the ring R.

Logarithmic derivations

Let $R \to A$ be a map of commutative rings. Recall that

$$\mathrm{Der}_R(A, J) := \{d \colon A \to J \mid d(ab) = ad(b) + bd(a)\}.$$

These are corepresented by the module of *Kähler differentials* $\Omega^1_{A|R}$:

$$\mathrm{Der}_R(A, J) \cong \mathrm{Hom}_{\mathrm{Mod}_A}(\Omega^1_{A|R}, J).$$

Explicit descriptions of $\Omega^1_{A|R}$ include

$$\Omega^1_{A|R} \cong A\{da\}/\!\sim \,\cong I/I^2,$$

where $I := \ker(A \otimes_R A \to A)$. This describes $\Omega^1_{A|R}$ as the conormal of the diagonal map

$$\mathrm{Spec}(A) \to \mathrm{Spec}(A \otimes_R A) \cong \mathrm{Spec}(A) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(A).$$

Logarithmic derivations

Let $(f, f^\flat)\colon (R, P, \beta) \to (A, M, \alpha)$ be a map of pre-log rings.

Definition

A *logarithmic derivation* $(d, d^\flat) \in \mathrm{Der}_{(R,P)}((A, M), J)$ consists of

- $d \in \mathrm{Der}_R(A, J)$
- $d^\flat\colon M \to (J, +)$

Logarithmic derivations

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Definition

A *logarithmic derivation* $(d, d^\flat) \in \mathrm{Der}_{(R,P)}((A, M), J)$ consists of

- $d \in \mathrm{Der}_R(A, J)$ such that $d(\alpha(m)) = \alpha(m) d^\flat(m)$
- $d^\flat\colon M \to (J, +)$

Logarithmic derivations

Let $(f, f^\flat)\colon (R, P, \beta) \to (A, M, \alpha)$ be a map of pre-log rings.

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A *logarithmic derivation* $(d, d^\flat) \in \mathrm{Der}_{(R,P)}((A, M), J)$ consists of

- $d \in \mathrm{Der}_R(A, J)$ such that $d(\alpha(m)) = \alpha(m)d^\flat(m)$
- $d^\flat\colon M \to (J, +)$ $d^\flat(f^\flat(p)) = 0.$

Suppose that $P = \{1\}$, so that part of the definition is simply a map of commutative monoids $d^\flat\colon M \to (J, +)$. We notice that

$$\mathrm{Hom}_{\mathrm{CMon}}(M, (J, +)) \cong \mathrm{Hom}_{\mathrm{Ab}}(M^{\mathrm{gp}}, (J, +)) \cong \mathrm{Hom}_{\mathrm{Mod}_A}(A \otimes_{\mathbb{Z}} M^{\mathrm{gp}}, J).$$

Fact

$\mathrm{Der}_{(R,P)}((A, M), J)$ *is corepresented by*

$$\Omega^1_{(A,M)|(R,P)} := \frac{\Omega^1_{A|R} \oplus (A \otimes_{\mathbb{Z}} (M^{\mathrm{gp}}/P^{\mathrm{gp}}))}{d\alpha(m) = \alpha(m) \otimes [m]}$$

Logarithmic derivations

There is a notion of logarithmic derivations $\mathrm{Der}_{(R,P)}((A, M), J)$, and this is corepresented by

$$\Omega^1_{(A,M)|(R,P)} := \frac{\Omega^1_{A|R} \oplus (A \otimes_{\mathbb{Z}} (M^{\mathrm{gp}}/P^{\mathrm{gp}}))}{d\alpha(m) = \alpha(m) \otimes [m]}.$$

We write $\mathrm{dlog}(m)$ for $1 \otimes [m]$, so that the defining relation reads $d\alpha(m) = \alpha(m)\mathrm{dlog}(m)$.

Example

There is a map of pre-log rings $(\mathbb{Z}_{(3)}, \langle 3 \rangle) \to (\mathbb{Z}_{(3)}[\sqrt{3}], \langle \sqrt{3} \rangle)$. For this map we have $\Omega^1_{(A,M)|(R,P)} \cong 0$, despite $\Omega^1_{A|R}$ being non-trivial.

More on logarithmic differentials

Example

There is a short exact sequence

$$0 \to \Omega^1_{\mathbb{Z}[x]} \to \Omega^1_{(\mathbb{Z}[x],\langle x\rangle)} \to \mathbb{Z} \to 0$$

isomorphic to

$$0 \to \mathbb{Z}[x]\{dx\} \to \mathbb{Z}[x]\{\mathrm{dlog}(x)\} \to \mathbb{Z} \to 0$$

sending dx to $dx = x\mathrm{dlog}(x)$. Using base-change and transitivity properties of the (log) Kähler differentials, this gives rise to sequences

$$0 \to \Omega^1_A \to \Omega^1_{(A,\langle a\rangle)} \to A/(a) \to 0$$

for any non-zero divisor $a \in A$.

Repletion

Goal: Describe $\Omega^1_{(A,M)|(R,P)} \cong I/I^2$ for some $I := \ker(B \to A)$. (For $B = A \otimes_R A$, this is precisely the Kähler differentials.)

Definition (Kato–Saito, Rognes)

$K \to M$ map of commutative monoids such that $K^{\mathrm{gp}} \to M^{\mathrm{gp}}$ is surjective. The *repletion* $K^{\mathrm{rep}} \to M$ by the pullback square

$$\begin{array}{ccc} K^{\mathrm{rep}} & \longrightarrow & K^{\mathrm{gp}} \\ \downarrow & & \downarrow \\ M & \longrightarrow & M^{\mathrm{gp}}. \end{array}$$

The universal property of the pullback induces a map $K \to K^{\mathrm{rep}}$.

In the same way that the coproduct $A \otimes_R A$ is relevant for describing the Kähler differentials, might suspect that the coproduct $M \oplus_P M$ is relevant for the log Kähler differentials.

Repletion examples

- Consider the addition map $+\colon \mathbb{N} \oplus \mathbb{N} \to \mathbb{N}$. In this case $(\mathbb{N} \oplus \mathbb{N})^{\mathrm{rep}} \to \mathbb{N}$ fits in a pullback square

$$\begin{array}{ccc} (\mathbb{N} \oplus \mathbb{N})^{\mathrm{rep}} & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} \\ \downarrow & & \downarrow \\ \mathbb{N} & \longrightarrow & \mathbb{Z}. \end{array}$$

  Isomorphic to $\mathbb{N} \oplus \mathbb{Z}$. The map from $\mathbb{N} \oplus \mathbb{N}$ sends $(n, m) \mapsto (n + m, m)$.
- More generally, we have $(M \oplus_P M)^{\mathrm{rep}} \cong M \oplus M^{\mathrm{gp}}/P^{\mathrm{gp}}$.

We recognize the quotient of group completions from the definition of log Kähler differentials.

The log diagonal

- Recall that $\Omega^1_{A|R} \cong I/I^2$, with $I := \ker(A \otimes_R A \to A)$.
- For $(R, P) \to (A, M)$, the coproduct $A \otimes_R A$ participates in a pre-log structure $M \oplus_P M \to (A \otimes_R A, \cdot)$: This is adjoint to $\mathbb{Z}[M \oplus_P M] \cong \mathbb{Z}[M] \otimes_{\mathbb{Z}[P]} \mathbb{Z}[M] \to A \otimes_R A$.

Theorem (Kato–Saito)

There is an isomorphism $\Omega^1_{(A,M)|(R,P)} \cong I/I^2$, *where*

$$I := \ker((A \otimes_R A) \otimes_{\mathbb{Z}[M \oplus_P M]} \mathbb{Z}[(M \oplus_P M)^{\mathrm{rep}}] \to A \otimes_{\mathbb{Z}[M]} \mathbb{Z}[M] \cong A).$$

This describes the log Kähler differentials as the conormal of the *log diagonal* in the sense of Kato–Saito.

Summary log structures

- A pre-log structure (R, P, β) consists of a map of commutative monoids $\beta\colon P \to (R, \cdot)$.
- Can be intepreted as a localization in-between R and $R[P^{-1}] := R \otimes_{\mathbb{Z}[P]} \mathbb{Z}[P^{\mathrm{gp}}]$.
- There is a notion of logarithmic derivations. Corepresented by a module $\Omega^1_{(A,M)|(R,P)}$ that arises as the conormal of a “log diagonal" map.
- This can be described in terms of the repletion $(M \oplus_P M)^{\mathrm{rep}}$.

$R \to A$ map of commutative rings.

Definition

The *Hochschild homology* $\mathrm{HH}^R(A)$ is the simplicial commutative R-algebra $S^1 \otimes^{\mathbb{L}}_R A$.

- The map $S^1 \to *$ induces $\mathrm{HH}^R(A) \to A$.
- The equivalence $S^1 \simeq * \sqcup_{*\sqcup *} *$ induces $\mathrm{HH}^R(A) \simeq A \otimes^{\mathbb{L}}_{A \otimes^{\mathbb{L}}_R A} A$.
  Derived self-intersections of the diagonal.

So Hochschild homology fits in a homotopy pushout

$$\begin{array}{ccc} A \otimes^{\mathbb{L}}_R A & \longrightarrow & A \\ \downarrow & & \downarrow \\ A & \longrightarrow & \mathrm{HH}^R(A) \end{array} \quad \xrightarrow{I/I^2} \quad \begin{array}{ccc} \mathbb{L}_{A|R} & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathbb{L}_{A|R}[1]. \end{array}$$

Here $\mathbb{L}_{A|R}$ is the *cotangent complex* of A relative to R.

Properties of the cotangent complex

$$\begin{array}{ccc} A \otimes_R^{\mathbb{L}} A & \longrightarrow & A \\ \downarrow & & \downarrow \\ A & \longrightarrow & \mathrm{HH}^R(A) \end{array} \quad \xrightarrow{I/I^2} \quad \begin{array}{ccc} \mathbb{L}_{A|R} & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathbb{L}_{A|R}[1]. \end{array}$$

- Classically, $\mathbb{L}_{A|R} := A \otimes_{A_\bullet} \Omega^1_{A_\bullet|R}$, $A_\bullet \to A$ free R-algebra resolution.
- If $R \to A$ is étale, $\mathbb{L}_{A|R} \simeq *$.
- If $R \to A$ is smooth, $\mathbb{L}_{A|R} \simeq \Omega^1_{A|R}[0]$.

Quillen establishes a spectral sequence

$$E^2_{p,q} = \pi_p(\Lambda^q_A \mathbb{L}_{A|R}) \implies \pi_{p+q}\mathrm{HH}^R(A).$$

Use this to show:

- If $R \to A$ étale, then $A \otimes_R \mathrm{HH}(R) \xrightarrow{\simeq} \mathrm{HH}(A)$.
- If $R \to A$ is smooth, $\Omega^*_{A|R} \xrightarrow{\cong} \pi_*\mathrm{HH}^R(A)$ iso of graded rings.

Logarithmic Hochschild homology

Let $(R,P) \to (A,M)$ be a map of pre-log rings. The Hochschild homology $\mathrm{HH}^R(A) := S^1 \otimes_R^{\mathbb{L}} A$ participates in a (simplicial) pre-log structure

$$M \oplus^{\mathbb{L}}_{M \oplus^{\mathbb{L}}_P M} M \simeq S^1 \oplus^{\mathbb{L}}_P M \to (S^1 \otimes^{\mathbb{L}}_R A, \cdot) =: (\mathrm{HH}^R(A), \cdot)$$

adjoint to

$$\mathbb{Z}[S^1 \oplus^{\mathbb{L}}_P M] \cong S^1 \otimes^{\mathbb{L}}_{\mathbb{Z}[P]} \mathbb{Z}[M] =: \mathrm{HH}^{\mathbb{Z}[P]}(\mathbb{Z}[M]) \to \mathrm{HH}^R(A).$$

Definition (Rognes)

The *log Hochschild homology* $\mathrm{HH}^{(R,P)}(A,M)$ is given by the homotopy pushout of

$$\mathrm{HH}^R(A) \leftarrow \mathrm{HH}^{\mathbb{Z}[P]}(\mathbb{Z}[M]) \cong \mathbb{Z}[S^1 \oplus^{\mathbb{L}}_P M] \to \mathbb{Z}[(S^1 \oplus^{\mathbb{L}}_P M)^{\mathrm{rep}}],$$

where the repletion is taken with respect to $S^1 \oplus^{\mathbb{L}}_P M \to M$.

Logarithmic Hochschild homology

Log Hochschild homology $\mathrm{HH}^{(R,P)}(A, M)$ is the homotopy pushout

$$\mathrm{HH}^R(A) \leftarrow \mathrm{HH}^{\mathbb{Z}[P]}(\mathbb{Z}[M]) \cong \mathbb{Z}[S^1 \oplus_P^{\mathbb{L}} M] \to \mathbb{Z}[(S^1 \oplus_P^{\mathbb{L}} M)^{\mathrm{rep}}].$$

Example (Rognes–Sagave–Schlichtkrull)

Recall that there are short exact sequences

$$0 \to \Omega^1_A \to \Omega^1_{(A,\langle a \rangle)} \to A/(a) \to 0$$

for $a \in A$ a non-zero divisor. Similarly, there are homotopy cofiber sequences

$$\mathrm{HH}(A) \to \mathrm{HH}(A, \langle a \rangle) \to \mathrm{HH}(A/(a))[1]$$

which recovers the former sequence on π_1.

Logarithmic Hochschild homology

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Recall that $\mathrm{HH}^R(A) \simeq A \otimes^{\mathbb{L}}_{A \otimes^{\mathbb{L}}_R A} A$, and

$$\mathbb{Z}[S^1 \oplus_P^{\mathbb{L}} M] \simeq \mathbb{Z}[M \oplus^{\mathbb{L}}_{M \oplus^{\mathbb{L}}_P M} M] \cong \mathbb{Z}[M] \otimes^{\mathbb{L}}_{\mathbb{Z}[M \oplus^{\mathbb{L}}_P M]} \mathbb{Z}[M].$$

Lemma

There is an equivalence

$$\mathbb{Z}[(S^1 \oplus_P^{\mathbb{L}} M)^{\mathrm{rep}}] \simeq \mathbb{Z}[M \bigoplus^{\mathbb{L}}_{(M \oplus^{\mathbb{L}}_P M)^{\mathrm{rep}}} M] \cong \mathbb{Z}[M] \bigotimes^{\mathbb{L}}_{\mathbb{Z}[(M \oplus^{\mathbb{L}}_P M)^{\mathrm{rep}}]} \mathbb{Z}[M].$$

So log Hochschild homology is the homotopy pushout of

$$\mathrm{HH}^R(A) \leftarrow \mathrm{HH}^{\mathbb{Z}[P]}(\mathbb{Z}[M]) \to \mathbb{Z}[(S^1 \oplus_P^{\mathbb{L}} M)^{\mathrm{rep}}].$$

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So log Hochschild homology is the homotopy pushout of

$$A \otimes_{\mathbb{Z}[M]} \mathbb{Z}[M] \leftarrow (A \otimes^{\mathbb{L}}_R A) \otimes_{\mathbb{Z}[M \oplus^{\mathbb{L}}_P M]} \mathbb{Z}[(M \oplus^{\mathbb{L}}_P M)^{\mathrm{rep}}] \to A \otimes_{\mathbb{Z}[M]} \mathbb{Z}[M].$$

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Log Hochschild homology $\mathrm{HH}^{(R,P)}(A, M)$ is the homotopy pushout

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There is an equivalence

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So log Hochschild homology is the homotopy pushout of

$$A \leftarrow (A \otimes^{\mathbb{L}}_R A) \otimes_{\mathbb{Z}[M \oplus^{\mathbb{L}}_P M]} \mathbb{Z}[(M \oplus^{\mathbb{L}}_P M)^{\mathrm{rep}}] \to A.$$

The log cotangent complex

Log Hochschild homology is the homotopy pushout of

$$A \leftarrow (A \otimes_R^{\mathbb{L}} A) \otimes_{\mathbb{Z}[M \oplus_P^{\mathbb{L}} M]} \mathbb{Z}[(M \oplus_P^{\mathbb{L}} M)^{\mathrm{rep}}] \rightarrow A,$$

the derived self-intersections of the log diagonal. In analogy with the picture

$$\begin{array}{ccc} A \otimes_R^{\mathbb{L}} A & \longrightarrow & A \\ \downarrow & & \downarrow \\ A & \longrightarrow & \mathrm{HH}^R(A) \end{array} \xrightarrow{I/I^2} \begin{array}{ccc} \mathbb{L}_{A|R} & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathbb{L}_{A|R}[1]. \end{array}$$

for Hochschild homology, we obtain

$$\begin{array}{ccc} (A \otimes_R^{\mathbb{L}} A) \otimes_{\mathbb{Z}[M \oplus_P^{\mathbb{L}} M]} \mathbb{Z}[(M \oplus_P^{\mathbb{L}} M)^{\mathrm{rep}}] & \longrightarrow & A \\ \downarrow & & \downarrow \\ A & \longrightarrow & \mathrm{HH}^{(R,P)}(A, M) \end{array} \xrightarrow{I/I^2} \begin{array}{ccc} \mathbb{L}_{(A,M)|(R,P)} & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathbb{L}_{(A,M)|(R,P)}[1]. \end{array}$$

Here $\mathbb{L}_{(A,M)|(R,P)}$ is the *log cotangent complex*.

Log étaleness and smoothness

Classically, $\mathbb{L}_{(A,M)|(R,P)} := A \otimes_{A_\bullet} \Omega^1_{(A_\bullet, M_\bullet)|(R,P)}$ for a free (R,P)-algebra resolution $(A_\bullet, M_\bullet) \to (A, M)$ (Gabber).

Definition (Slightly informal)

We say that $(R,P) \to (A,M)$ is

- *log étale* if $R \otimes_{\mathbb{Z}[P]} \mathbb{Z}[M] \to A$ is étale and $|M^{\mathrm{gp}}/P^{\mathrm{gp}}| \in \mathrm{GL}_1(A)$.
- *log smooth* if $R \otimes_{\mathbb{Z}[P]} \mathbb{Z}[M] \to A$ is smooth and $M^{\mathrm{gp}}/P^{\mathrm{gp}}$ finitely generated with torsion of order invertible in A.

Lemma

Suppose that $\mathbb{Z}[P] \to \mathbb{Z}[M]$ is flat. Then

- $\mathbb{L}_{(A,M)|(R,P)} \simeq *$ *if* $(R,P) \to (A,M)$ *log étale;*
- $\mathbb{L}_{(A,M)|(R,P)} \simeq \Omega^1_{(A,M)|(R,P)}[0]$ *if* $(R,P) \to (A,M)$ *log smooth.*

Use sequence $A \otimes^{\mathbb{L}}_{\mathbb{Z}} (M^{\mathrm{gp}}/P^{\mathrm{gp}}) \to \mathbb{L}_{(A,M)|(R,P)} \to \mathbb{L}_{A|R \otimes_{\mathbb{Z}[P]} \mathbb{Z}[M]}$.

Log étaleness and smoothness

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Example

$(\mathbb{Z}_{(3)}, \langle 3 \rangle) \to (\mathbb{Z}_{(3)}[\sqrt{3}], \langle \sqrt{3} \rangle)$ is log étale but not étale.

Example

Weird but important pre-log structure: $\langle t \rangle \to (\mathbb{Z}, \cdot)$ sending $t^0 \mapsto 1$, $t^n \mapsto 0$. Adjoint to $\mathbb{Z}[t] \to \mathbb{Z}$ sending t to 0. Map of pre-log rings

$$(\mathbb{Z}, \langle t \rangle) \xrightarrow{(i,\Delta)} (\mathbb{Z}[x, y]/(xy), \langle x \rangle \oplus \langle y \rangle)$$

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$$(\mathbb{Z}, \langle t \rangle) \xrightarrow{(i,\Delta)} (\mathbb{Z} \otimes_{\mathbb{Z}[\langle t \rangle]} \mathbb{Z}[\langle x \rangle \oplus \langle y \rangle], \langle x \rangle \oplus \langle y \rangle)$$

with map of monoids the diagonal. Log smooth but not smooth.

Log étaleness and smoothness

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- *log smooth* if $R \otimes_{\mathbb{Z}[P]} \mathbb{Z}[M] \to A$ is smooth and $M^{\mathrm{gp}}/P^{\mathrm{gp}}$ finitely generated with torsion of order invertible in A.

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$(\mathbb{Z}_{(3)}, \langle 3 \rangle) \to (\mathbb{Z}_{(3)}[\sqrt{3}], \langle \sqrt{3} \rangle)$ is log étale but not étale.

Example

Weird but important pre-log structure: $\langle t \rangle \to (\mathbb{Z}, \cdot)$ sending $t^0 \mapsto 1$, $t^n \mapsto 0$. Adjoint to $\mathbb{Z}[t] \to \mathbb{Z}$ sending t to 0. Map of pre-log rings

$$(\mathbb{Z}, \langle t \rangle) \xrightarrow{(i,\Delta)} (\mathbb{Z} \otimes_{\mathbb{Z}[t]} \mathbb{Z}[x, y], \langle x \rangle \oplus \langle y \rangle)$$

with map of monoids the diagonal. Log smooth but not smooth.

A logarithmic HKR-theorem

The new description of logarithmic Hochschild homology gives a spectral sequence

$$E^2_{p,q} = \pi_p(\bigwedge_A^q \mathbb{L}_{(A,M)|(R,P)}) \implies \pi_{p+q}\mathrm{HH}^{(R,P)}(A, M).$$

Key ingredient in proving

Theorem (Binda–Park–L.–Østvær)

Suppose $\mathbb{Z}[P] \to \mathbb{Z}[M]$ *is flat. Then*

- $A \otimes_R \mathrm{HH}(R, P) \to \mathrm{HH}(A, M)$ *is an equivalence if* $(R, P) \to (A, M)$ *log étale;*
- $\Omega^*_{(A,M)|(R,P)} \to \pi_*\mathrm{HH}^{(R,P)}(A, M)$ *iso of graded rings if* $(R, P) \to (A, M)$ *is log smooth.*

The flatness hypothesis becomes redundant once everything in sight is made *dividing Nisnevich* local (Binda–Park–Østvær).

Topological Hochschild homology

Let $R \to A$ be a map of commutative ring spectra. Define $\mathrm{THH}^R(A) := S^1 \otimes_R A$ (implicitly derived). As before there is a homotopy pushout

$$\begin{array}{ccc} A \otimes_R A & \longrightarrow & A \\ \downarrow & & \downarrow \\ A & \longrightarrow & \mathrm{THH}^R(A) \end{array} \xrightarrow{I/I^2} \begin{array}{ccc} \mathbb{L}_{A|R} & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathbb{L}_{A|R}[1]. \end{array}$$

Here $\mathbb{L}_{A|R}$ is the *topological André–Quillen homology* of A relative to R. This is often denoted by $\mathrm{TAQ}^R(A)$.

Topological logarithmic structures

Definition (First guess)

A *pre-log ring spectrum* (R, P, β) consists of

- a commutative ring spectrum R;
- an $\mathbb{E}_\infty$-space P;
- a map $P \to \Omega^\infty(R)$; equivalently a map $\mathbb{S}[P] := \Sigma^\infty_+ P \to R$.

Problem: This notion will not give rise to pre-log structures $(R, \langle x \rangle)$ satisfying $R[\langle x \rangle^{-1}] \simeq R[x^{-1}]$ for x a homotopy class in strictly positive degree.

Example

We would like a pre-log ring spectrum $(\mathrm{ku}, \langle u \rangle)$ with $\mathrm{ku}[\langle u \rangle^{-1}] \simeq \mathrm{KU}$. This does not exist with the above definition (Rognes).

Topological logarithmic structures

Solution (Sagave–Schlichtkrull): Consider instead $\mathbb{E}_\infty$-spaces with a $QS^0 := \Omega^\infty\Sigma^\infty S^0$-grading. This comes with a $(\mathbb{S}_*, \Omega^\infty_*)$-adjunction, a good theory of group completion, repletion...

Definition

A pre-log ring spectrum (R, P, β) consists of

- a commutative ring spectrum R;
- a QS^0-graded $\mathbb{E}_\infty$-space P;
- a map $P \to \Omega^\infty_*(R)$; equivalently a map $\mathbb{S}_*[P] \to R$.

Example (Sagave–Schlichtkrull)

There is a pre-log ring spectrum $(\mathrm{ku}, \langle u\rangle_*)$ with $\mathrm{ku}[\langle u\rangle_*^{-1}] \simeq \mathrm{KU}$.

Log topological Hochschild homology

If $(R, P) \to (A, M)$ is a map of pre-log ring spectra, $\mathrm{THH}^R(A)$ participates in a pre-log structure $S^1 \oplus_P M \to \Omega^\infty_*(S^1 \otimes_R A)$ adjoint to

$$\mathbb{S}_*[S^1 \oplus_P M] \cong S^1 \otimes_{\mathbb{S}_*[P]} \mathbb{S}_*[M] \to S^1 \otimes_R A =: \mathrm{THH}^R(A).$$

Definition (Rognes–Sagave–Schlichtkrull)

The *log topological Hochschild homology* $\mathrm{THH}^{(R,P)}(A, M)$ is the homotopy pushout of

$$\mathrm{THH}^R(A) \leftarrow \mathrm{THH}^{\mathbb{S}_*[P]}(\mathbb{S}_*[M]) \cong \mathbb{S}_*[S^1 \oplus_P M] \to \mathbb{S}_*[(S^1 \oplus_P M)^{\mathrm{rep}}],$$

where the repletion is taken with respect to $S^1 \oplus_P M \to M$.

Localization sequences in log THH

Log THH is the homotopy pushout of

$$\mathrm{THH}^R(A) \leftarrow \mathrm{THH}^{\mathbb{S}_*[P]}(\mathbb{S}_*[M]) \cong \mathbb{S}_*[S^1 \oplus_P M] \to \mathbb{S}_*[(S^1 \oplus_P M)^{\mathrm{rep}}].$$

Recall that there are homotopy cofiber sequences

$$\mathrm{HH}(A) \to \mathrm{HH}(A, \langle a \rangle) \to \mathrm{HH}(A/(a))[1]$$

in ordinary log Hochschild homology.

Theorem (Rognes–Sagave–Schlichtkrull)

There is a homotopy cofiber sequence

$$\mathrm{THH}(\mathrm{ku}) \to \mathrm{THH}(\mathrm{ku}, \langle u \rangle_*) \to \Sigma \mathrm{THH}(\mathbb{Z}).$$

Tempting to think of this as a THH-version of Blumberg–Mandell's

$$K(\mathrm{ku}) \to K(\mathrm{KU}) \to \Sigma K(\mathbb{Z}),$$

but not so clear-cut: for ko we have

$$\mathrm{THH}(\mathrm{ko}) \to \mathrm{THH}(\mathrm{ko}, \langle \beta \rangle_*) \to \Sigma \mathrm{THH}(\tau_{\leq 7}\mathrm{ko}).$$

The log cotangent complex

A version of log TAQ is studied by Rognes–Sagave. From the definitions it is not clear that this relates to log THH in the manner we would expect from the linear case.

Theorem (L.)

This all works out.

More precisely, there is a homotopy pushout

$$\begin{array}{ccc} (A \otimes_R A) \otimes_{\mathbb{S}_*[M \oplus_P M]} \mathbb{S}_*[(M \oplus_P M)^{\mathrm{rep}}] & \longrightarrow & A \\ \downarrow & & \downarrow \\ A & \longrightarrow & \mathrm{THH}^{(R,P)}(A, M) \end{array} \xrightarrow{I/I^2} \begin{array}{ccc} \mathbb{L}_{(A,M)|(R,P)} & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathbb{L}_{(A,M)|(R,P)}[1] \end{array}$$

Moreover, $\mathbb{L}_{(A,M)|(R,P)}$ agrees with previous notions of log TAQ. This fits into Lurie's cotangent complex formalism.

Log étale maps

Definition (Slightly informal)

A map $(R, P) \to (A, M)$ of pre-log ring spectra is

- *log étale* if $R \otimes_{\mathbb{S}_*[P]} \mathbb{S}_*[M] \to A$ is étale and $A \otimes (M^{\mathrm{gp}}/P^{\mathrm{gp}}) \simeq *$;
- *log smooth*? (Krause–Nikolaus' forthcoming spherical HKR-theorem will provide some hints.)

Let $\ell_p \to \mathrm{ku}_p$ denote the inclusion of the p-complete Adams summand. On coefficient rings, $\mathbb{Z}_p[v_1] \to \mathbb{Z}_p[u]$ sends v_1 to u^{p-1}.

Theorem (Sagave, reformulated)

The map $(\ell_p, \langle v_1 \rangle_*) \to (\mathrm{ku}_p, \langle u \rangle_*)$ *of pre-log ring spectra is log étale.*

As a consequence, the log cotangent complex is contractible:

$$A \otimes (M^{\mathrm{gp}}/P^{\mathrm{gp}}) \to \mathbb{L}_{(A,M)|(R,P)} \to \mathbb{L}_{A|R\otimes_{\mathbb{S}_*[P]}\mathbb{S}_*[M]}.$$

Log étale maps

Theorem (L.)

- *$(R, P) \to (A, M)$ log étale of discrete pre-log rings induces log étale map of pre-log ring spectra provided that $\mathbb{Z}[P] \to \mathbb{Z}[M]$ is flat.*
- *$(R, P) \to (A, M)$ log étale of pre-log ring spectra implies that*

$$A \otimes_R \mathrm{THH}(R, P) \to \mathrm{THH}(A, M)$$

is an equivalence.

We get examples of base-change in log THH:

- $\mathbb{Z}_{(3)}[\sqrt{3}] \otimes_{\mathbb{Z}_{(3)}} \mathrm{THH}(\mathbb{Z}_{(3)}, \langle 3 \rangle) \xrightarrow{\simeq} \mathrm{THH}(\mathbb{Z}_{(3)}[\sqrt{3}], \langle \sqrt{3} \rangle)$
- $\mathrm{ku}_p \otimes_{\ell_p} \mathrm{THH}(\ell_p, \langle v_1 \rangle_*) \xrightarrow{\simeq} \mathrm{THH}(\mathrm{ku}_p, \langle u \rangle_*)$.

The latter recovers a theorem by Rognes–Sagave–Schlichtkrull. Heuristically, this exhibits the inclusion of the Adams summand as a tamely ramified extension.

Log étale maps

The complexification map $\mathrm{ko} \to \mathrm{ku}$ participates in a map

$$(\mathrm{ko}, \langle \beta \rangle_*) \to (\mathrm{ku}, \langle u \rangle_*)$$

of pre-log ring spectra.

Theorem (Höning–Richter)

The log cotangent complex $\mathbb{L}_{(\mathrm{ku}, \langle u \rangle_*)|(\mathrm{ko}, \langle \beta \rangle_*)}$ is not contractible.

This implies that $(\mathrm{ko}, \langle \beta \rangle_*) \to (\mathrm{ku}, \langle u \rangle_*)$ is not log étale. Heuristically, this means that the complexification map if wildly ramified.

Things to do

- Stronger HKR-statements in characteristic zero (Toën–Vezzosi for ordinary HH).
- More examples: There is evidence that $\mathrm{tmf}_0(3) \to \mathrm{tmf}_1(3)$ is wildly ramified. Good log structures with two generators?
- Ramification at chromatic primes? (Example: $\mathrm{ku}_p \to \mathrm{THH}^{\ell_p}(\mathrm{ku}_p)$ a $K(1)$-local equivalence [Angeltveit].)
- Relationship to algebraic K-theory? (Probably very difficult.)
- One idea: Bhatt–Morrow–Scholze style filtration on log TC. (Related to forthcoming work of Bhatt–Clausen–Mathew).

Thanks for listening!